

# Statistical Physics for Humanities: A Tutorial

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The image of physics is connected with simple “mechanical” deterministic events: that an apple always falls down, that force equals mass times acceleration. Indeed, applications of such concept to social or historical problems go back two centuries (population growth and stabilisation, by Malthus and by Verhulst) and use “differential equations”, as recently reviewed by Vitanov and Ausloos [2011]. However, since even today’s computers cannot follow the motion of all air molecules within one cubic centimeter, the probabilistic approach has become fashionable since Ludwig Boltzmann invented Statistical Physics in the 19th century. Computer simulations in Statistical Physics deal with single particles, a method called agent-based modelling in fields which adopted it later. Particularly simple are binary models where each particle has only two choices, called spin up and spin down by physicists, bit zero and bit one by computer scientists, and voters for the Republicans or for the Democrats in American politics (where one human is simulated as one particle). Neighbouring particles may influence each other, and the Ising model of 1925 is the best-studied example of such models. This text will explain to the reader how to program the Ising model on a square lattice (in Fortran language); starting from there the readers can build their own computer programs. Some applications of Statistical Physics outside the natural sciences will be listed.

## 1 Introduction

*Learning by Doing* is the intention of this tutorial: readers should learn how to construct their own models and to program them, not learn about the great works of the author [Stauffer et al 2006] and the lesser works of his competitors [Billari et al 2006].

Already Empedokles is reported to have 25 centuries ago compared humans to fluids: Some are easy to mix, like wine and water; and some, like oil and water, refuse to mix. Newspapers can give you recent human examples, and physicists and others have taken up the challenge to study selected problems of history and other humanities [Castellano et al 2009] with methods similar to physics. In the opposite direction, Prados [2009] uses the physics dream of “unified field theory” to describe his history of the Vietnam war.

The next section will recommend ways how to construct models, the following one how to program a simple Ising model on a square lattice, and a concluding section will list some applications. An appendix will introduce the Fortran programming language.

## 2 Model Building

### 2.1 What is a Model?

“Models” in physics and in this tutorial usually deal with the single elements of a system and how their interactions produce the behaviour of the whole system. Outside of physics, a model may just be any mathematical law approximating reality. Thus the statement that human adult mortality increases exponentially with age is often called the Gompertz model by demographers but the Gompertz rule by physicists; the latter ones use the Penna model of individuals undergoing genetic mutations and Darwinian selection to simulate a large population perhaps obeying Gompertz [Stauffer et al 2006].

### 2.2 Binary versus more complicated models

If no previous work on a general model is known, I recommend to start with binary variables where each particle has only two choices, called spin up and spin down by physicists, bit zero and bit one by computer scientists, occupied or empty for percolation, and voters for the Republicans or for the Democrats in American politics (where one human is simulated as one particle). Of course reality is more complicated, but we want to *understand* reality: Does it agree with the simplest possible model? (Simulations for pilot training etc. are different [Bridson and Batty 2010]). Thus we follow the opinion of Albert Einstein that a model should be as simple as possible, but not simpler. If you want to simulate traffic jams [Chowdhury et al. 2000] in cities, the colour of the cars is quite irrelevant, but for visibility in the dark the colour matters.

Once the binary case has been studied, one can go to more than two choices. If three groups are fighting each other, obviously three choices are needed in a model [Lim 2007]. Even in a two-party political system like the USA, other candidates were important in Florida for the US presidential elections of 2000. The opinions of people [Malarz et al 2011] are in reality continuous and can be modelled by one or several real numbers between zero and unity, or between minus infinity and plus infinity. Nevertheless it is standard practice in opinion polls to allow only a few choices like full agreement, partial agreement, neutrality, partial disagreement, full disagreement. And in elections one can only vote among the discrete number of candidates or parties which are on the ballot. In the Kosovo opinion of the International Court of Justice (July 22, 2010) one judge criticised the binary tradition of either legal or illegal, stating that tolerable is in between; nevertheless the court majority stayed with the binary logic of not illegal.

Physicists like to call the binary variables “spins” but readers from outside physics should refrain from studying the quite complicated spin concept in quantum mechanics. Spins are simply up and down (1 or 0; 1 or  $-1$ ). Similarly, don’t be deterred if physicist talk about a Hamiltonian; in most cases this is just the energy from high school.

## 2.3 Humans are neither Spins nor Atoms

Of course, that is true, but it does not exclude that humans are modelled like spins. Modern medicine blurred the boundary between life and death; nevertheless we usually talk about people having been born in a certain year, and having died in another year, as if they were binary up-down variables. Reasons for death are complicated and I don't even know mine yet; nevertheless demographers like Gompertz estimated probabilities for dying at some age. Such probabilities are rather useless for predicting the death of one individual, but averaged over many people they may give quite accurate results. When I throw one coin I do not know how it will fall; when I throw thousand coins, usually about half of them fall on one side and the others on the other side (law of large numbers). If I throw 1000 coins and all show "head", most likely I cheated. Thus to simulate one person's opinion and decisions on a computer does not seem to be realistic; to do the same for millions of people may give good average properties, like the number of deaths at the age of 80 to 81 years, or the fraction of voters selecting the parties in an upcoming election. The whole insurance industry is based on this law of large numbers. Humans are not spins but many humans together might be studied well by spin models.

## 2.4 Deterministic or Statistical ?

Non-physicists often believe that physics deals with deterministic rules: The apple falls down from the tree and not up; force equals mass times acceleration; etc. (Or they have heard of quantum-mechanical probability and apply that to large systems where such quantum effects should be negligibly small.) In this sense the cause of World War I was seen as a consequence of the arms race modelled by deterministic differential equations for averages [Richardson 1935]. And the decay of empires was described [Geiss 2008] as starting at the geographical periphery, since the influence from the center decreases towards zero with increasing distance, just as the gravitational force between the sun and its planets; see also [Diamond 1997, epilog].

Because of its historical importance let us look into Richardson's papers of 1935: Two opposing (groups of) nations change their preparedness  $x$  for war because of three reasons: 1) the war preparedness of the other side; 2) fatigue and expense; 3) dissatisfaction with existing peace treaties. Reasons 1 and 3 increase and reason 2 decreases the war preparedness; reason 1 is proportional to the  $x$  of the opponent, reason 2 to the own  $x$ , and reason 3 independent of  $x_1$  and  $x_2$ . Even complete disarmament ( $x_1 = x_2 = 0$ ) at one moment does not help if there is dissatisfaction with the existing peace. And if reason 1 is stronger than reason 2 for both sides, both  $x$  increase exponentially with time towards infinity. His second paper details the mathematical solutions for these linear coupled inhomogenous differential equations.

More recently, the widespread use of computers shifted the emphasis to more realistic *probabilistic* models, using random numbers to simulate the throwing of coins or other statistical methods. This "Statistical Physics" and a simple

example are the subject of the next section.

## 3 Statistical Physics and the Ising Model

### 3.1 Boltzmann Distribution

No present computer can simulate the motion of all air molecules in a cubic centimeter. Fortunately, Ludwig Boltzmann about 150 years ago invented a simple rule. The molecules move at temperature  $T$  with a velocity which can change all the time but follows a statistical distribution: The probability for a velocity  $v$  is proportional to  $\exp(-E/T)$ , where  $E$  is the kinetic energy of the molecule due to its velocity. The same principle applied to a binary choice, where a particle can be in two states A and B with energies  $E_A$  and  $E_B$  means that the two probabilities are

$$p_A = \frac{1}{Z} \exp(-E_A/T); \quad p_B = \frac{1}{Z} \exp(-E_B/T); \quad (1a)$$

$$Z = \exp(-E_A/T) + \exp(-E_B/T) \quad (1b)$$

since the sum over all probabilities must be unity. More generally, a configuration with energy  $E$  is in thermal equilibrium found with probability

$$p = \frac{1}{Z} \exp(-E/T); \quad Z = \sum \exp(-E/T) \quad (2)$$

where the sum runs over all possible states of the system.  $Z$  is called the partition function, one of the rare cases where the German word for it, Zustandssumme = sum over states, is clearer and shorter. The temperature  $T$  is measured neither in Celsius (centigrade) nor Fahrenheit but  $T = 0$  at the absolute zero temperature (about  $-273$  Celsius below the freezing temperature of water) and moreover is measured in energy units. (If  $T$  is measured in Kelvin, the corresponding energy is  $k_B T$  where  $k_B$  is the Boltzmann constant and set to unity in the present tutorial.) The function  $\exp(x)$  is the exponential function, also written as  $e^x$ , which for integer  $x$  means the product of  $x$  factors  $e \simeq 2.71828$ ;  $e^x = 2^{x/0.69315} = 10^{0.4343x}$ .

Eqs.(1,2) can be regarded as axioms on which Statistical Physics is built, like the Parallel Axiom of Euclidean Geometry, but in some cases they can be derived from other principles. Humanities are allowed to use these ideas of Boltzmann since history institutes were named after him.

### 3.2 Ising Model

In 1925, Ernst Ising (born in the heart of the city this author lives in) finished his doctoral dissertation on a model for ferromagnetism, which became famous two decades later and was shown to apply to liquid-vapour transitions half a century later. We assume that each site of a lattice (e.g. a square lattice where each site  $i$  has four neighbours: clockwise up, right, down, left) carries a spin

$S_i = \pm 1$  (up or down). Neighbouring spins “want” to be parallel, i.e. they have an energy  $-J$  if they are in the same state and an energy  $+J$  if they are in the two different states. Moreover, a “magnetic” field  $H$  between minus infinity and plus infinity (also called  $B$ ) tries to orient the spins in its own direction. The total energy then is

$$E = -J \sum_{\langle ij \rangle} S_i S_j - H \sum_i S_i \quad (3)$$

where the first sum goes over all ordered pairs of neighbor sites  $i$  and  $j$ . Thus the “bond” between sites  $A$  and  $B$  appears only once in this sum, and not twice (for  $i = A, j = B$  as well as for  $i = B, j = A$ ). The second sum runs over all sites of the system. Thus  $2J$  is the energy to break one bond, and  $2H$  is the energy to flip a spin from the direction of the field into the opposite direction. As discussed before in the Boltzmann subsection, the higher the energy  $E$  is the lower is the probability to observe this spin configuration; at infinitely high temperatures  $T$  all configurations are equally probable; at  $T = 0$  all spins must be parallel to each other and to the field  $H$  in equilibrium. The “magnetisation”  $M$  is the number of up spins minus the number of down spins,

$$M = \sum_i S_i \quad . \quad (4)$$

Computer simulations of this Ising model will be described in the appendix.

Applied to human beings, this Ising model could represent two possible opinions in a population; everybody tries to convince the neighbours of the own opinion ( $J$ ), and in addition the government  $H$  may try to convince the whole population of its own opinion. The temperature then gives the tendency of the individuals not to think like the majority of their neighbours and the government. Zero temperature thus means complete conformity, and infinite temperature completely random opinions.

Theories with paper and pencil in two dimensions as well as computer simulations give  $M(T, H)$ . In particular, for  $H = 0$  and in more than one dimension, the equilibrium magnetisation  $M = \pm M_0$  is a non-zero spontaneous magnetisation for  $T < T_c$  and is zero for  $T \geq T_c$  where  $T_c$  is the critical or Curie temperature (named after Pierre Curie, not his more famous wife Marie Curie). On the square lattice,  $T_c/J \simeq 2.27$  is known exactly, in three dimensions only numerically; in one dimension there is no transition to a spontaneous magnetisation, as Ernst Ising had shown:  $T_c = 0$ .

The above model obeys Isaac Newton’s law *actio* = *-reactio*: The sun attracts the earth with the same force as the earth attracts the sun, only in opposite direction. Human relations can also be unsymmetrical: He loves her but she does not love him. Then the bond between sites  $i$  and  $j$  may be directed instead of the usual case of an undirected bond. For example, if  $i$  influences  $j$  but  $j$  does not influence  $i$ , flipping the spin at  $j$  from parallel ( $S_j = +S_i$ ) to antiparallel ( $S_j = -S_i$ ) to the spin at  $i$  can cost an energy  $2J$  while flipping  $S_i$  at constant  $S_j$  costs nothing. In this case no unique energy  $E(S_i, S_j)$  is defined

for this spin pair and thus such models have been much less studied in the physics literature. (In this example, one could gain a lot of energy from nothing by the cyclic process of flipping  $j$  from antiparallel to parallel, gaining energy  $2J$ , then flipping  $i$  from parallel to antiparallel, costing nothing, then flipping  $j$  again and so on. Such a perpetuum mobile does not exist in physics.)

Outside physics, of course, one can commit such crimes against energy conservation and forget all probabilities proportional to  $\exp(-E/T)$ . Instead one can assume arbitrary probabilities, as long as their sum equals one. For example, if an element has three possible states A, B and C, then one may assume that with probability  $p$  an A becomes B, a B becomes C, and a C becomes A; with probability  $1 - p$  the element does not change, and there is no backward process from C to B to A to C. Then one has a circular perpetuum mobile, which may be realistic for some social processes. Physicists sometimes distinguish between dynamics (when the changes are determined fully by energy or force) and kinetics (when additional assumptions, like the probabilities of eqs.(1) are made); probabilities independent of energy/force are then kinetics, as is most of the material described here. And usually to find a stationary or static equilibrium, one has to wait for many non-equilibrium iterations in the simulation (in a static situation, nothing moves anymore; in a stationary simulation the averages are nearly constant since changes in one direction are mostly cancelled by changes of other elements in the opposite direction. There are many choices, one needs not mathematics, but mathematical thinking: precise and step-by-step.

### 3.3 Warning against Mean Field Approximation

*This subsection contains many formulas and can be skipped.* If you want to get answers by paper and pencil, you can use the mean field approximation (also called molecular field approximation), which in economics corresponds to the approximation by representative agent. Approximate in the first sum of Eq.(3) the  $S_j$  by its average value, which is just the normalised magnetisation  $m = M/L^2 = \sum_i S_i/L^2$ . Then the energy is

$$E = -J \sum_{\langle ij \rangle} S_i m - H \sum_i S_i = -H_{eff} \sum_i S_i$$

with the effective field

$$H_{eff} = H + \sum_j m = H + qm$$

where the latter sum runs over the  $q$  neighbours only and is proportional to the magnetization  $m$ . Thus the energy  $E_i$  of spin  $i$  no longer is coupled to other spins  $j$  and equals  $\pm H_{eff}$ . The probabilities  $p$  for up and down orientations, according to Eq.(1), are now

$$p(S_i = +1) = \frac{1}{Z} \exp(H_{eff}/T); \quad p(S_i = -1) = \frac{1}{Z} \exp(-H_{eff}/T)$$

and thus

$$m = p(S_i = +1) - p(S_i = -1) = \tanh(H_{eff}/T) = \tanh[(H + qm)/T]$$

with the function  $\tanh(x) = (e^x - e^{-x})/(e^x + e^{-x})$ . This implicit equation can be solved graphically; for small  $m$  and  $H/T$ ,  $\tanh(x) = x - x^3/3 + \dots$  gives

$$H/T = (1 - T_c/T)m + \frac{1}{3}m^3 + \dots; \quad T_c = qJ$$

related to Lev Davidovich Landau’s theory of 1937 for critical phenomena ( $T$  near  $T_c$ ,  $m$  and  $H/T$  small) near phase transitions.

All this looks very nice except that it is wrong: In the one-dimensional Ising model,  $T_c$  is zero instead of the mean field approximation  $T_c = qJ$ . The larger the number of neighbours and the dimensionality of the lattice is, the more accurate is the mean field approximation. Basically, the approximation to replace  $S_i S_j$  by an average  $S_i m$  takes into account the influence of  $S_j$  on  $S_i$  but not the fact that this  $S_i$  again influences  $S_j$  creating a feedback. If the mathematics of this subsection looks deterrent, just ignore it; you are recommended to use computer simulations of single interacting spins, and not mean field theories. Outside of physics such simulations are often called “agent based” [Billari et al 2006, Bonabeau 2002]; presumably the first one was the Metropolis algorithm published in 1953 by the group of Edward Teller, who is historically known from the US hydrogen bomb and Strategic Defense Initiative (Star Wars, SDI).

## 4 Applications

### 4.1 Schelling Model for Social Segregation

Economics nobel laureate Thomas C. Schelling advised the US government on war and peace in the 1960s and published a methodologically crucial paper (cited more than 500 times a decade later) [Schelling 1971, Fossett 2011, Henry et al 2011] which introduced methods of statistical physics to sociology. Each Ising spin corresponds to one of two ethnic groups (black and white in big US cities) both of which prefer not to be surrounded by the other group. The above Ising model then shows that for  $T < T_c$  segregation emerges without any outside control: The simulated region becomes mostly white or mostly black. Unfortunately, Schelling simulated a more complicated version of the Ising lattice at  $T = 0$  which did not give large “ghettos” like Harlem in New York City, but only small clusters of predominantly white and black residences. Only by additional randomness [Jones 1985] (cited only 8 times, mostly by physicists) can “infinitely” large ghettos appear. It is easier to just simulate the above Ising model [Sumour et al 2008, 2011], which also allows people of one group to move into another city (or away from the simulated region) to be replaced by residents of the other group. Sumour et al also cite earlier Schelling-type simulations by physicists. For nearly three decades physicists ignored the

Schelling model; now the sociologists ignore the physics simulations of the last decade and the much earlier Jones [1985] paper.

As examples we give two iIsing-model figures from Müller et al [2008] where people also increase their amount  $T$  of tolerance = social temperature if they see that their whole neighbourhood belongs to the same group as they themselves. And afterwards they slowly forget and reduce their tolerance. With changing forgetting rate one may either observe lots of small clusters, Fig.1, or one big ghetto, Fig.2.

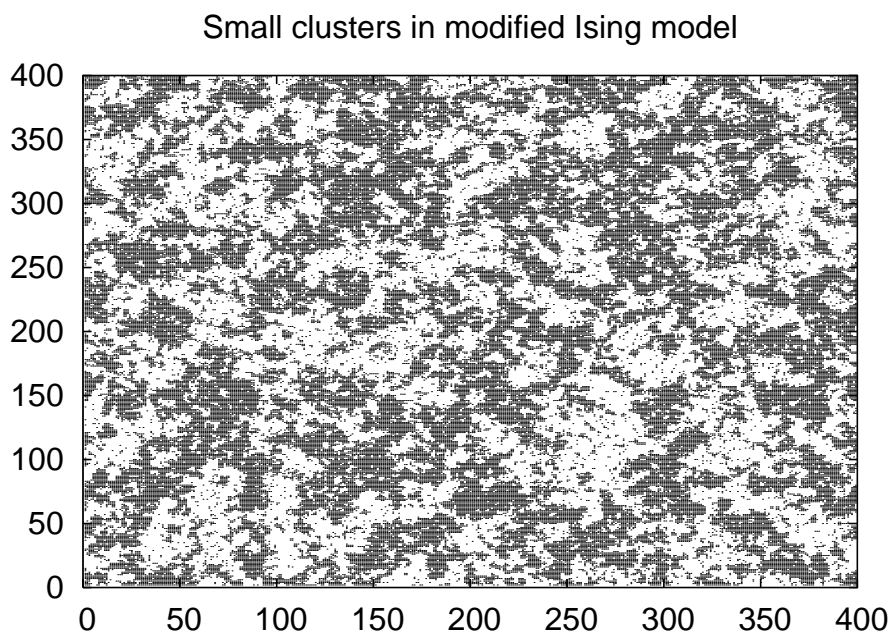


Figure 1: Cluster formation for slow forgetting in the Ising modification of Müller et al [2008]; no large ghetto is formed.

About simultaneously with this Schelling paper, physicist Weidlich started his sociodynamics approach to apply the style of physics to social questions [Weidlich 2000], see also [Galam 2008].

## 4.2 Sociophysics and Networks

A good overall review of the sociophysics field (of which the Schelling model is just one example) was given by Castellano et al [2009]. Of particular interest is the reproduction of universal properties of election results with many candidates: The curves of how many candidates got  $n$  votes each are similar to each



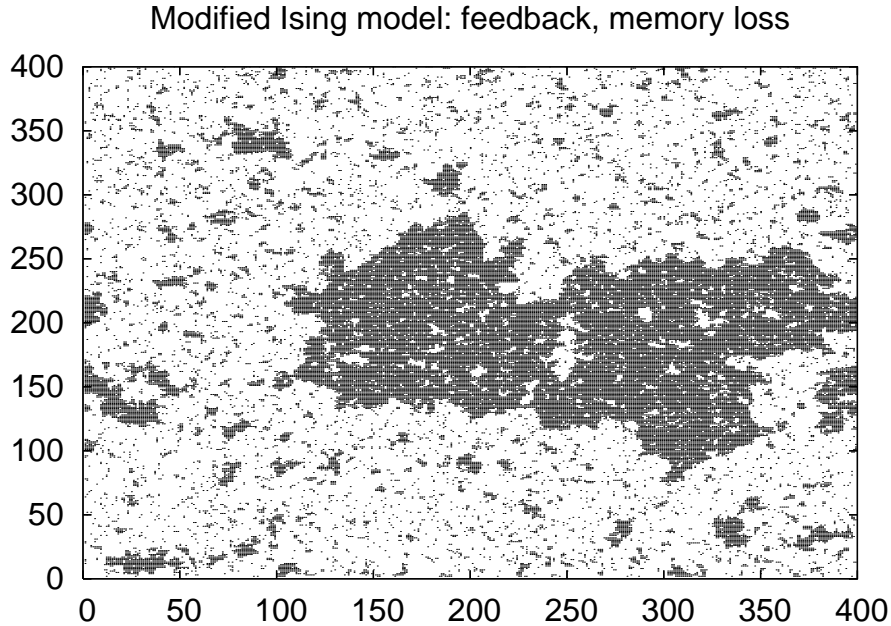


Figure 2: In the same model of the previous figure, a large ghetto is formed if people forget faster their learned tolerance.

other [Castellano 2009]. Car traffic [Chowdhury et al [2000], economic markets [Bouchaud and Potters 2010, Bonabeau 2002], opinion dynamics [Malarz et al 2011], stone-age culture [Shennan 2001], histophysics [Lam 2002], languages [Schulze et al 2008], Napoleon’s decision before the battle of Waterloo [Mongin 2008], religion [Ausloos and Petroni 2009], political secession from a state [Lustick 2011], demography on social networks [Fent et al. 2011], insurgent wars in Iraq and Afghanistan [Johnson et al. 1944], ... are other applications. For example, non-physicists [Holman et al 2011] have acknowledged that physicists Serva and Petroni (2008) may have been the first to use Levinstein distances to calculate the ages of language groups. (These distances are differences between words for the same meaning in different languages.)

Students in class may sit on a square lattice, but normally humans do not. (In universities, mostly only a part of the lattice is occupied, which physicists simulate as a “dilute” square lattice.) They may be connected by friendship or job not only with nearest neighbors but also with people further away. Such networks, investigated for a long time by sociologists [Stegbauer and Haeussling 2010], were studied by physicists intensively for a dozen years [Barabási 2002, Albert and Barabási 2002, Bornholdt and Schuster 2003, Cohen and Havlin

2010]. In the Watts-Strogatz (or “small world”) network a random fraction of nearest-neighbour bonds is replaced by bonds with sites further away, selected randomly from the whole lattice. If that fraction approaches unity, one obtains the Erdős-Rényi networks, a limit of percolation theory [Flory 1941]. More realistic are the scale-free Barabási-Albert networks, where the network starts with a small core and then each newly added site forms a bond with a randomly selected already existing member of the network. The selection probability is proportional to the number of bonds which the old member already has acquired: Famous people get more attention and more “friends” than others; no lattice is assumed here anymore. In all these networks, the average number of bonds needed to connect two randomly selected sites increases logarithmically with the number of sites in the network, whereas for  $d$ -dimensional lattices this average number of bonds increases stronger with a power law, exponent  $1/d$ . Having simulated one network, one can also study connected sets of networks or other social networks [Watts, Dodds, Newman 2002], or demography on them [Fent et al 2011].

The latest application is Statistical Justice: In May 2011, John Demjanjuk was sentenced for having helped in 1943 in the murder of more than 28,000 Dutch Jews in the Nazi concentration camp of Sobibór. One knows who was deported but not who survived the transport from the Netherlands to Poland. And one does not know which duties the accused had there on which day. Thus all acts of the camp guards were regarded as having helped in their murder, and the number near 28,000 was estimated from the average death rate during the transports. The verdict thus gave neither the name of a murder victim nor the day of a murder, but was entirely based on statistical averages. [Times 2011].

## 5 Appendix: How to Program the Ising Model

The following Fortran manual and program are both short and should encourage the reader to learn this technique.

### 5.1 Fortran Manual

Fortran = formula translator an early language (above machine code or assembler) for computer programming; many others followed and in particular C++ is widespread, but nevertheless this tutorial uses Fortran which is closer to plain English and allows to easily find a typical programming error (using an array outside its defined bounds). If your Fortran program is called `name.f`, it can be compiled with `f95 -O name.f` (or 77 instead of 95; 0 = optimisation), and executed with `./a.out` (or just `a.out`). The just mentioned error message appears when using `f95 -fbounds-check name.f ; ./a.out`, but execution then is much slower and thus instead -O should be used after error correction.

Fortran commands usually start in column 7 and end before column 73. A C in column 1 signifies a comment for the reader, to be ignored by the computer. In column 6 we write a 1 if this line is a continuation of the previous line,

while columns 2 to 5 are reserved for labels, i.e. numbers to control the flow of commands. For example, `GOTO 7` means to jump to the line labelled by 7.

Variable names start with a letter; names starting with I, J, K, L, M, N signify integers without rounding errors, other names are real (floating-point) numbers and nearly always have rounding errors. The operations `+`, `-`, `*`, `/` and `SQRT`, `COS`, `SIN`, `EXP` etc have their usual meaning except that `N/M` is always rounded downwards to an integer value; e.g. `3/5` is zero. Also `I = X` means rounding downwards. Since the natural logarithm is not an integer it is denoted by `ALOG` instead of `log`.

Decisions are made automatically, e.g.

```
IF(A.GT.0) B = SQRT(A)
```

where `.GT.` means greater than, with analogous meanings for `.LT.`, `.GE.`, `.LE.`, `.EQ.`, `.NE.`, `.NOT.`, `.AND.`, `.OR.`.

A loop is executed by

```
DO 99 K=M,N
```

which means that all lines from this line down to and including the line with label 99 are executed for  $k = m, m + 1, m + 2, \dots, n$ . One may put inner loops into outer loops, if needed.

Arrays need to be declared at the beginning of a program, for example through

```
DIMENSION A(100,100), B(100), C(L)
```

Here, if  $C$  has an arbitrary dimension  $L$ , then  $L$  must be given a value before this dimension statement through

```
PARAMETER(L = 100)
```

and must not be changed throughout the program. Similarly, variables can be initialised via a data line like

```
DATA L/100/, B/100*1.0/
```

but only once at the beginning of the program, not later again.

Results are best printed out through

```
PRINT *, x, y, z
```

Thereafter execution should stop with a `STOP` line, followed by an `END` line.

The statement

```
n = n+1
```

is not an equality (which then could be simplified to the nonsensical  $0 = 1$ ) but a command to the computer: to find the place in the memory where the variable `n` is stored, to get the value of `n` from there, to add one to it, and to store the sum in that same memory place as the new value for `n`. Some computer languages therefore use `:=` instead of the simpler but misleading `=` sign.

Normally it does not matter whether or not CAPITAL letters are used. The computer language Basic is rather similar to Fortran. Now we bring a complete program to simulate the Ising model on the square lattice.

## 5.2 Ising Model Program

```
c      heat bath 2D Ising in a field
```

```

parameter(L=1001,Lmax=(L+2)*L)
dimension is(Lmax),ex(-4:4)
data t,mcstep,iseed/0.90,1000,1/,h/+0.50/,ex/9*0.0/
print *, '#', L,mcstep,iseed,t,h
x=rand(iseed)
Lp1=L+1
LspL=L*L+L
L2p1=2*L+1
do 1 i=1,Lmax
1   is(i)=1
do 2 ie=-4,4,2
    x=exp(-ie*2.0*0.440688/t-h)
2   ex(ie)=x/(1.0+x)
do 3 mc=1,mcstep
    mag=0
    do 4 i=Lp1,LspL
c      if(i.ne.L2p1) goto 6
c      do 5 j=1,L
c5      is(j+LspL)=is(j+L)
6      ie=is(i-1)+is(i+1)+is(i-L)+is(i+L)
        is(i)=1
        if(rand().lt.ex(ie)) is(i)=-1
4      mag=mag+is(i)
c      do 7 i=1,L
c7      is(i)=is(i+L*L)
3      if(mc.eq.(mc/100)*100) print *, mc, mag
    stop
end

```

The parameter line fixes the size of the  $L \times L$  square; the sites in it are numbered by one index, typewriter style. Thus the right neighbor of site  $2L$  is  $2L + 1$  and sits on the left end of the next line: Helical boundary conditions. The lower neighbour of site  $2L$  is  $2L + L$ , the upper neighbour is  $2L - L$ , and the left neighbor is  $2L - 1$ . The spins in the top and bottom buffer lines ( $1 \dots L$  and  $L^2 + L + 1 \dots L^2 + 2L$ ) stay in their initial up orientation; if instead one wants periodic boundary conditions in the vertical direction (better to reduce boundary influence), one has to omit the five comment symbols C at and before loops 5 and 7.

The temperature enters the data line in units of  $T_c$ , i.e.  $T = 0.9T_c$  in this example; the known value  $J/T_c = 0.44068 \dots$  is used nine lines later. In the same line the field, in units of  $k_B T$ , is given as 0.5. In the line after the first print statement, `rand(iseed)` initialises the random number generator (see next subsection for warning and improvement); a different seed integer gives different random numbers. Later `rand()` produces the next “random” number between 0 and 1 from the last one in a reproducible but hardly predictable way. Loop 2 determines the Boltzmann probabilities `ex` needed for Eqs.(1) and finishes the initialisation. Loop 3 makes `mcstep` iterations (time steps = Monte Carlo steps per spin). Loop 4 runs over all spins except those in the two buffer lines and

determines the local interaction energy  $ie$  as the sum over the four neighbour spins. Then we set the spin to  $+1$ , and if the conditions of Eqs(1) so require, instead it is set to  $-1$ . (Generally, a command is executed with a probability  $p$  if the random number `rand()` is smaller than  $p$ .) We print out the magnetisation only every hundred time steps in order to avoid too much data on the computer screen. The results are plotted in Fig.3 and show that the temperature is very low: Even though it is only 10 percent below the critical temperature  $T_c$ , the magnetisation barely changes from its initial value 1002001 and is after 100 iterations already in equilibrium, also due to the applied field. In three instead of two dimensions, without a field, and at temperatures closer to  $T_c$  longer times are needed for equilibration.

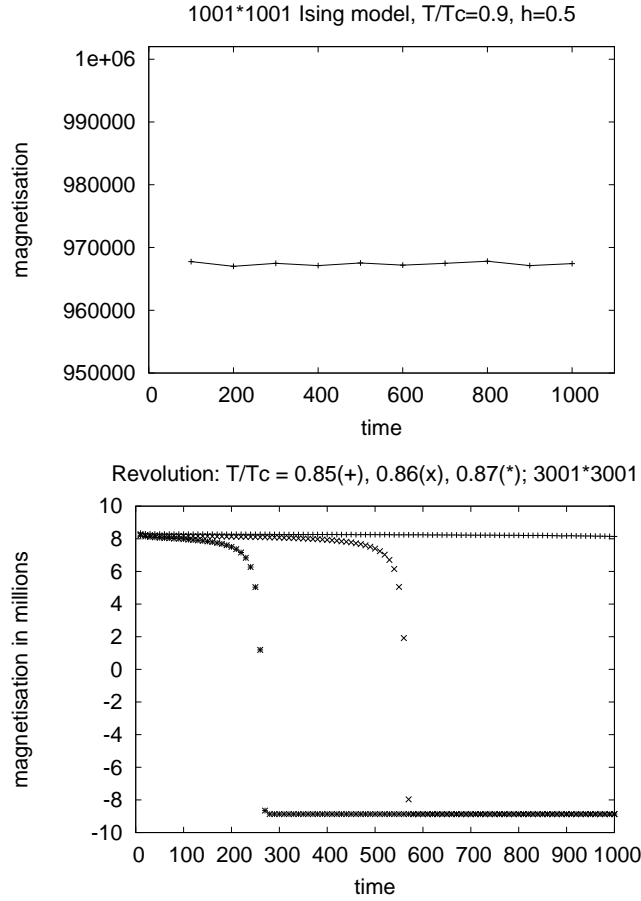


Figure 3: Top: Results from the listed Ising program. Bottom: Revolutions

Perhaps you find it more interesting to simulate revolutions, as in Fig.1b, where the field  $h$  was equal to the fraction of overturned spins (suggestion of

Sorin Solomon for Bornholdt-type model). So, what is difficult about computations? Do you know a shorter Fortran manual?

### 5.3 Random Numbers

The above `rand` produces random numbers in an easily programmed way, but often this may be slow and/or bad, or the used algorithm is unknown to the user. It is better to program random number generation explicitly. If you multiply by hand two nine-digit integers, you may easily predict the first and the last digit of the product, but hardly the digits in the middle, except by tediously doing the whole multiplication correctly. Similarly, if `ibm` is a 32-bit odd integer, then the product

```
ibm=ibm*16807
```

is again an odd integer, and normally requires 46 bits. (A bit = binary digit is a zero or one in a computer.) The computer throws away the leading bits and leaves the least significant 32 bits. The first bit gives the sign, thus plus times plus gives minus in about half the cases, in contrast to what you learned in elementary school. The last of these remaining bits is predictably always set to one (odd integers) but the leading (most significant) bits are quite random. (Actually, your computer may do something very similar when you call `rand()`.) More precisely, they are pseudo-random; in order to search for errors one wants to get exactly the same random numbers when one repeats a simulation with the same seed.

These random 32-bit integers `ibm` between  $-2^{31} - 1$  and  $+2^{31} - 1$  can be transformed into real numbers through `ran=factor*ibm+0.5` where `factor = 0.5/2147483647`, but it is more efficient to normalize the probabilities  $p$  (here =  $\text{ex}/(1.0+\text{ex})$ ) to the full interval of 32-bit integers through  $(2.0*p-1.0)*2147483647$ , once at the beginning of the simulation. Then the next random integers `ibm` simply have to be compared with this normalized probability. In the above Ising program, one then stores the Boltzmann probabilities as `dimension iex(-4:4)` at the beginning, calculates them through

```
2 iex(ie)=(2.0*ex/(1.0+ex) - 1.0)*2147483647
```

in the initialisation, and later merely needs

```
ibm=ibm*16807
```

```
if(ibm.lt.iex(ie)) is(i)=-1
```

in the above Ising program.

With 32 bits the pseudo-random integers are repeated after  $2^{29}$  such multiplications with 16807, which is a rather small number for today's personal computers. It is better to use 64 bits via

```
integer*8 ibm, iex
```

at the beginning of the program in order to get many more different random numbers, using e.g.

```
2 iex(ie)=2147483647*(4.0*ex/(1.0+ex) - 2.0)*2147483647
```

for the normalized 64-bit probabilities. Now the quality is much better without much loss in speed; unfortunately one now can make much more programming errors involving these random numbers.

## 5.4 History

Now comes a list of extended abstracts, a few pages each, about what this author finds interesting in recent history, available on request from dstauff@thp.uni-koeln.de:

Who is to blame for World War I  
No miracle on the Marne, 9/9/1914  
Lies and Art. 231 of Versailles Peace Treaty 1919  
Was Hitler's 1941 attack against the Soviet Union a preemptive war ?  
Had Hitler nearly gotten Mosnow in 1941 ?  
Why was there no joint Japanese attack when Hitler attacked the Soviet Union  
The Sea Battle of Leyte, 25 October 1944  
Did Soviet tanks approach Tehran in March 1946 ?  
Missed chance for peace in Korea, October 1950?  
Stalin's proposal of March 1952 for a united Germany  
1956: West German finger on the nuclear trigger ?  
Tank confrontation at Checkpoint Charlie 10/1961  
The 1962 Cuban Missile crisis: security or prestige?  
Lyndon B. Johnson (1908-1973) and the Dominican crisis (US Invasion 1965)  
1990: East Germany into NATO ?  
Kosovo War 1999  
The start of the Libyan war, March 2011

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Albert R. and Barabási A.L. [2002], "Statistical mechanics of complex networks", *Reviews of Modern Physics* **74**, 47-97.

Ausloos M. and Petroni F., "Statistical dynamics of religion evolutions", *Physica A* **388**, 4438-4444; M. Ausloos [2010], "On religion and language evolutions seen through mathematical and agent based models", in *Proceedings of the First Interdisciplinary CHES Interactions Conference*, C. Rangacharyulu and E. Haven, Eds., World Scientific, Singapore, pp. 157-182.

Barabási A.L. [2002] "Linked", Perseus, Cambridge.

Billari F.C., Fent T., Prskawetz A., and Scheffran J. [2006] *Agent-based computational modelling*, Physica-Verlag, Heidelberg.

Bonabeau E. [2002] "Agent-based modelling: Methods and techniques for simulating human systems", *Proc. Natl. Acad. Sci. USA* **99**, 7280-7287.

Bornholdt S. and Schuster H.G. [2003] *Handbook of graphs and networks*, Wiley-VCH, Weinheim.

Bouchaud J.P. and Potters M. [2009] *Theory of financial risks and derivative pricing*, Cambridge University Press, Cambridge.

- Bridson R. and Batty C [2010] “Computational physics in film”, *Science* **330**, 1756-1757.
- Castellano C., Fortunato S., Loreto V., [2009] “Statistical physics of social dynamics”, *Rev. Mod. Physics* **81**, 591-646.
- Chowdhury D., Santen L., Schadschneider A. [2000] “Statistical physics of vehicular traffic and some related systems”, *Physics Reports* **329**, 199-329.
- Cohen R. and Havlin S. [2010] *Complex Networks*, Cambridge University Press, Cambridge.
- Diamond, J [1997] *Guns, Germs, and Steel*, Norton, New York.
- Fent T., Diaz B.A, Prskawetz A. [2011] “Family policies in the context of low fertility and social structure”, *Vienna Inst. Demogr. Working Paper 2/2011* ([www.oeaw.at/vid](http://www.oeaw.at/vid)).
- Flory, P.J. [1941] “Molecular size distribution in three-dimensional polymers: I, II, III”, *J. Am. Chem. Soc.* **63**, 3083, 3091, 3096.
- Fossett, M. [2011] “Generative models of segregation”, *J. Math. Sociology* **35**, 114-145.
- Galam S. [2008] “Sociophysics: A review of Galam models”, *Int. J. Mod. Phys. C* **19**, 409-440.
- Geiss, I. [2008] *Geschichte im Überblick*, Anaconda, Köln.
- Henry, A.D., Pralat, P., Zhang, C-Q. [2011] *Proc. Natl. Acad. Sci. USA* **108**, 8505-8610.
- Holman, E.W., 14 coauthors [2011], “Automated dating of the world’s language families based on lexical similarity”, preprint for *Current Anthropology*.
- Johnson, N., Carran, S., Botner, J., Fontaine, K., Laxague, N., Nuetzel, P., Turnley, J., Tivnan, B. [2011] “Pattern in escalations in insurgent and terrorist activity”, *Science* **333**, 81-84.
- Jones F.L. [1985] “Segregation models of group segregation”, *Aust. New Zeal. J. Sociol.* **21**, 431-444.
- Lim M., Metzler R., Bar-Yam Y. [2007] “Global pattern formation and ethnic/cultural violence”, *Science* **317**, 1540-1544. See also Hadzibeganovic T. et al [2008], *Physica A* **387**, 3242-3252.
- Lustick J. [2011] “Secession of the center: A virtual probe of the prospects for Punjabi secessionism in Pakistan and the Secession of Punjabistan”, *Journal of Artificial Societies and Social Simulation* **14**, issue 1, paper 7 (electronic only via [jasss.soc.surrey.ac.uk](http://jasss.soc.surrey.ac.uk)).
- Malarz K., Gronek P. and Kułakowski K. [2011] “Zaller-Deffuant model of mass opinion”, *Journal of Artificial Societies and Social Simulation* **14**, issue 1, paper 2 (electronic only via [jasss.soc.surrey.ac.uk](http://jasss.soc.surrey.ac.uk)).



- Mongin P. [2008] “Retour à Waterloo - Histoire militaire et théorie des jeux”, *Annales. Histoire, Sciences Sociales* **63**, 39-69.
- Müller, K., Schulze, C., Stauffer, D. [2008] “Inhomogeneous and self-organized temperature in Schelling-Ising model”, *Int. J. Mod. Phys. C* **19**, 385-391.
- Prasos, J. [2009] “Vietnam”, University Press of Kansas, Lawrence 2009, p. xiii.
- Richardson L.F. [1935] “Mathematical psychology of war”, *Nature* **135**, 830-831 and **136**, 1025-1026.
- Schelling T.C. “Dynamic models of segregation” [1971] *J. Math. Sociol.* **1** 143-186.
- Schulze C., Stauffer D., and Wichmann S. [2008] “Birth, survival and death of languages by Monte Carlo simulation”. *Comm. Comput. Phys.* **3**, 271-294.
- Shennan S. [2001] “Demography and cultural innovation: a model and its implications for the emergence of modern human culture”, *Cambridge Archeol. J.* **11**, 5-16.
- Stauffer D., Moss de Oliveira S., de Oliveira P.M.C., Sá Martins J.S. [2006]. *Biology, sociology, geology by computational physicists*, Elsevier, Amsterdam.
- Stegbauer C. and Haeussling R. (eds.) [2010], *Handbuch Netzwerkforschung*, VS-Verlag, Wiesbaden.
- Sumour M.A., El-Astal, A.H., Radwan M.M., Shabat, M.M. [2008], Urban segregation with cheap and expensive residences, *Int. J. Mod. Phys. C* **19**, 637-645.
- Sumour M.A., Radwan M.M., Shabat, M.M. [2011], “Highly nonlinear Ising model and social segregation”, arXiv:1106.5574 (electronically only on arXiv.org section physics).
- Times: nytimes.com May 12, 2011 “Demjanjuk”.
- Vitanov, N.K. and Ausloos, M. R.[2011] “ Knowledge epidemics and population dynamics models for describing idea diffusion”. In: *Models of Science Dynamics - Encounters between Complexity Theory and Information Sciences*, ed. by Scharnhorst, A. Boerner, K., P. van den Besselaar. Springer, Berlin Heidelberg (forthcoming)
- Watts, D.J., Dodds, P.S., and Newman, M.E.J. [2002] “Identity and search in social networks”, *Science* **296**,1302-1305.
- Weidlich W. [2000] *Sociodynamics; a systematic approach to mathematical modelling in the social sciences* Harwood Academic Publishers; 2006 reprint: Dover, Mineola (New York).

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